Chapter 16: The Eady Cyclone Model

In middle latitudes most of the variations in our weather come from the travelling low and high pressure systems which typically travel across the country from west south-west to east north-east. These are observed to grow and decay, taking a few days to go through a life cycle. One source of energy for these waves is the potential energy implicit in the general increase in temperature towards the tropics. If a motion system begins to develop and it is such as to allow the warm air to rise and the cold air to sink, then the centre of mass of the whole atmosphere will descend, reducing the potential energy, which will presumably be converted into kinetic energy of the motion. The existence of the temperature gradients means that the flow is baroclinic. As some perturbations to the flow grow, the flow is clearly unstable to such disturbances. The motion is therefore said to the result of *baroclinic instability*.

The simplest model of this type of motion was introduced by Eady in 1949 and is hence known as the *Eady cyclone model*. We shall broadly follow his development in this chapter.

Eady treated the simplest case he could devise which would contain the physical processes which he expected to operate in real cyclones. He therefore treated the case of an incompressible fluid on an f -plane. The approximation of incompressibility may seem a little drastic, as the atmosphere is manifestly compressible. Two things might be said however. One is that the Eady treatment would be directly applicable to the case of the ocean and has merits on those grounds. The other is that subsequent investigators have generalised the solution to the compressible case and have shown that the basic results from the incompressible case carry over into the compressible case too. He also made drastic assumptions about the boundary conditions. The processes are assumed to be confined to the troposphere. The lower surface is taken to be a rigid plane at sea level. The upper boundary is assumed to be the tropopause and in his initial treatment this was taken to be a plane rigid lid too, mimicking the increase in static stability in the stratosphere which tends to suppress vertical motion there. In his original paper he did consider more realistic treatments which did have a stratosphere, and hence a better treatment of the tropopause, but these are beyond our scope.

In the last chapter we considered a compressible fluid on a β -plane. This had a potential vorticity equation

$$\left(\frac{\partial}{\partial t}+u_g\frac{\partial}{\partial x}+v_g\frac{\partial}{\partial y}\right)_p q=0$$

where

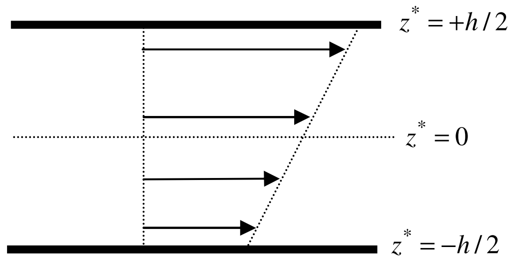
$$q=\nabla^2\psi+f+\frac{f_0^2}{N^2}\left(\frac{\partial^2\psi}{\partial z^{*2}}-\frac{1}{H}\frac{\partial\psi}{\partial z^*}\right)$$

To make this applicable to an f -plane is easy; all we need to do is replace f by f_0. It is a little more complicated to re-derive this formula for the incompressible case. However if H is thought of as the density scale height (which it is in an isothermal atmosphere), then in the case of an incompressible atmosphere the density does not change with height so H becomes infinite. Thus the potential vorticity becomes

$$q = \nabla^2\psi + f_o + \frac{f_0^2}{N^2}\left(\frac{\partial^2\psi}{\partial z^{*2}}\right)$$

A full analysis (beyond the scope of this course) confirms that this is the correct form.

Now the basic state in the model needs a temperature gradient to so that there is a source of warm and cold air. The simplest assumption is to have a temperature gradient which is uniform (in space and time) and directed equatorwards. The thermal wind relationship tells us that this will be accompanied by a uniform vertical windshear. It proves convenient to set the origin of the height co-ordinates to be midway between the ground and the tropopause. If the depth of the troposphere is h, we shall hence write the basic flow as $u_0 = u_{00} + \Lambda z^*$ and the boundary conditions will be imposed at $z^* = -h/2$ (the surface) and $z^* = +h/2$ the tropopause.



Linearizing the quasi-geostrophic equation gives

$$\left\{\frac{\partial}{\partial t} + \left(u_{00} + \Lambda z^*\right)\frac{\partial}{\partial x}\right\}\left\{\nabla^2\psi' + \frac{f_0^2}{N^2}\left(\frac{\partial}{\partial z^*}\right)^2\psi'\right\} = 0$$

On the face of it this equation has a complicated structure in the vertical, but we obtain a solution if the second bracket is zero. This simply says that the eddy quasi-geostrophic potential vorticity is zero, so that the total quasi-geostrophic potential vorticity is constant for a particle (and all particles) at f_0.

$$\nabla^2\psi' + \frac{f_0^2}{N^2}\left(\frac{\partial}{\partial z^*}\right)^2\psi' = 0$$

This has solutions of the form $\psi' \propto \exp i\left(\sigma t + \lambda x + \mu y + \nu z^*\right)$ provided that

$$-\left(\lambda^2 + \mu^2\right) - \frac{f_0^2}{N^2}\nu^2 = 0$$

i.e. that

$$\nu^2 = -\frac{N^2}{f_0^2}\left(\lambda^2 + \mu^2\right).$$

Now if the solution is wavelike in the horizontal the right hand side will be negative and ν will be wholly imaginary. So set $\nu = i\nu_i$ where ν_i is real. Then $\nu_i = \pm\alpha$ with

$$a \equiv \frac{N}{f_0}\sqrt{\left(\lambda^2 + \mu^2\right)}$$

Eq 1

so that α is the scaled total horizontal wavenumber.

Thus full solutions will look like linear combinations of components of the form

$$\psi' = \left(A \exp \alpha z^* + B \exp - \alpha z^*\right) \exp i\left(\sigma t + \lambda x + \mu y\right)$$

Eq 2

where A and B are constants to be determined from the boundary conditions and α is given by Eq 1.

So what boundary conditions do we need to impose? The simplest assumption for the lower surfaces is that it is a rigid plane, so that there can be no vertical velocity there. Hence we shall impose $w^* = 0$ at $z^* = -h/2$. It is not so clear what should be done at the tropopause but as the static stability of the stratosphere is very much larger than that of the tropopause, it will be difficult for motions in the troposphere to raise the tropopause. Hence as a first approximation we will assume that $w^* = 0$ at $z^* = h/2$ also[1]. To impose these boundary conditions, we obviously need an equation which relates the vertical velocity to the stream-function. The thermodynamic equation provides the link.

The thermodynamic equation in the incompressible case

In equation 5 of chapter 15 the relation between the streamfunction and temperature was shown to be $\dfrac{\partial \psi}{\partial z^*} = \dfrac{R}{f_0 H_{00}} T$. If we expand H_{00} we get

$$\frac{\partial \psi}{\partial z^*} = \frac{g}{f_0 T_{00}} T$$

and it turns out that this form is valid in the incompressible case too.

The thermodynamic equation (ch 15 eq 7 carries over almost unchanged

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)_p \frac{\partial \psi}{\partial z^*} + w^* N^2 = 0$$

[1] Eady treated this case, but he also treated other cases which are beyond our scope. For instance he considered a two-layer model similar to our treatment but with an infinite layer of high static stability above it, intended to represent the stratosphere.

the only difference being that we have the actual Brunt-Vaisaila frequency rather than its modified form.

Linearising this by writing $(u,v)=(u_0,0)+(u',v')$ and $\psi=\psi_0+\psi'$ gives

$$\left(\frac{\partial}{\partial t}+u_0\frac{\partial}{\partial x}\right)_p\frac{\partial\psi'}{\partial z^*}+\left(v'\frac{\partial}{\partial y}\right)_p\frac{\partial\psi_0}{\partial z^*}+w^{*'}N^2=0$$

$$\left(v'\frac{\partial}{\partial y}\right)_p\frac{\partial\psi_0}{\partial z^*}=v'\frac{\partial u_0}{\partial z^*}=\Lambda v'=\Lambda\frac{\partial\psi'}{\partial x}$$

$$\left(\frac{\partial}{\partial t}+u_0\frac{\partial}{\partial x}\right)_p\frac{\partial\psi'}{\partial z^*}+\Lambda\frac{\partial\psi'}{\partial x}+w^{*'}N^2=0$$

The boundary conditions

The statement that w^* is zero at $z^*=\pm h/2$ becomes the pair of statements

$$\left(\frac{\partial}{\partial t}+\frac{h\Lambda}{2}\frac{\partial}{\partial x}\right)_p\frac{\partial\psi'}{\partial z^*}+\Lambda\frac{\partial\psi'}{\partial x}=0 \text{ at } z^*=+h/2$$

and

$$\left(\frac{\partial}{\partial t}-\frac{h\Lambda}{2}\frac{\partial}{\partial x}\right)_p\frac{\partial\psi'}{\partial z^*}+\Lambda\frac{\partial\psi'}{\partial x}=0 \text{ at } z^*=-h/2$$

or, using relationships such as $\frac{\partial\psi'}{\partial x}=i\lambda\psi'$ for these wavelike solutions,

$$\left(\sigma+\frac{h\Lambda}{2}\lambda\right)_p\left.\frac{\partial\psi'}{\partial z^*}\right|_{z=h/2}+\Lambda\lambda\psi'\big|_{z=h/2}=0$$

$$\left(\sigma-\frac{h\Lambda}{2}\lambda\right)_p\left.\frac{\partial\psi'}{\partial z^*}\right|_{z=-h/2}+\Lambda\lambda\psi'\big|_{z=-h/2}=0$$

When we substitute for ψ' from Eq 2 we get a pair of simultaneous equations for A and B of the form

$$\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Eq 3

in which the elements r_{ij} involve σ and λ, μ (and hence α - see Eq 1). As Eq 3 is homogeneous, it can only be consistent provided the determinant of the matrix is zero, i.e.

$$\begin{vmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{vmatrix} = 0.$$

When this is expanded we find that

$$\left(\frac{\sigma}{\lambda} + u_{00}\right)^2 = \Lambda^2 h^2 \left[\frac{1}{4} - \frac{\coth(\alpha h)}{\alpha h} + \frac{1}{(\alpha h)^2}\right]$$

Eq 4

This equation relates the horizontal scale of the waves (encapsulated in α and λ) to their time variation (given by σ).

Growth rates and the shortwave cut-off

All the terms on the right hand side of Eq 4 are real, so it is real, but clearly can be positive or negative. Now as $\alpha h \to 0$, $\coth \alpha h \to \infty$, while as $\alpha h \to \infty$, $\coth \alpha h \to 1$. It follows that there is a critical value of α, which we will denote by α_c, such that when $\alpha \geq \alpha_c$, the right hand side is non-negative, and σ is wholly real, while if $\alpha < \alpha_c$ then σ has to have an imaginary part and the amplitude of the solution will grow or decay in time. As small α corresponds to large wavelengths, we deduce that waves shorter than a critical wavelength have constant amplitude, while waves longer than the critical value may grow or decay. We can say therefore that there is a shorwave cut-off to growth.

Neutral waves

It is convenient to write $\sigma = \sigma_r + i\sigma_i = -c\lambda + i\sigma_i$, where σ_r, σ_i and c are all real (and c is the phase speed of the wave), then for short waves (which have $\alpha \geq \alpha_c$) σ_i has to be zero and $(u_{00} - c) = \pm\sqrt{R.H.S}$ As our interest is mainly in growing disturbances we shall not pursue these neutral waves any further.

Growing waves

For longer waves (which have $\alpha < \alpha_c$), it is apparent by equating real and imaginary parts that

$$(u_{00} - c) = 0$$

and that

$$\left(\frac{\sigma_i}{\lambda}\right) = \pm\Lambda h\sqrt{\left|\frac{1}{4} - \frac{\coth(\alpha h)}{\alpha h} + \frac{1}{(\alpha h)^2}\right|}$$

Eq 5

.

The first of these two equations tells us that the growing waves move with the velocity of the middle level flow.

The amplitude of the waves varies in time as $\exp - \sigma_i t$, so the growing waves are found by selecting the negative sign in Eq 5.

Scale selection in growing waves

The rate at which waves grow is determined by the magnitude of $\lambda\Lambda h\sqrt{\left|\frac{1}{4} - \frac{\coth(\alpha h)}{\alpha h} + \frac{1}{(\alpha h)^2}\right|}$. For a given μ there will be a particular value of λ for which the waves grow faster than those with other values.

As an illustration of how this might work, we can first consider waves with phase lines orientated N-S, i.e. with $\mu = 0$. It turns out that this gives a maximum growth rate when $\alpha h \approx 1.6$. Now since $ah = \frac{Nh}{f_0}\sqrt{(\lambda^2 + \mu^2)} = \frac{Nh}{f_0}\lambda$ we have that for maximum growth, $1.6\frac{f_0}{Nh} = \lambda$ or $L_x = \frac{2\pi}{1.6}\frac{Nh}{f_0}$.

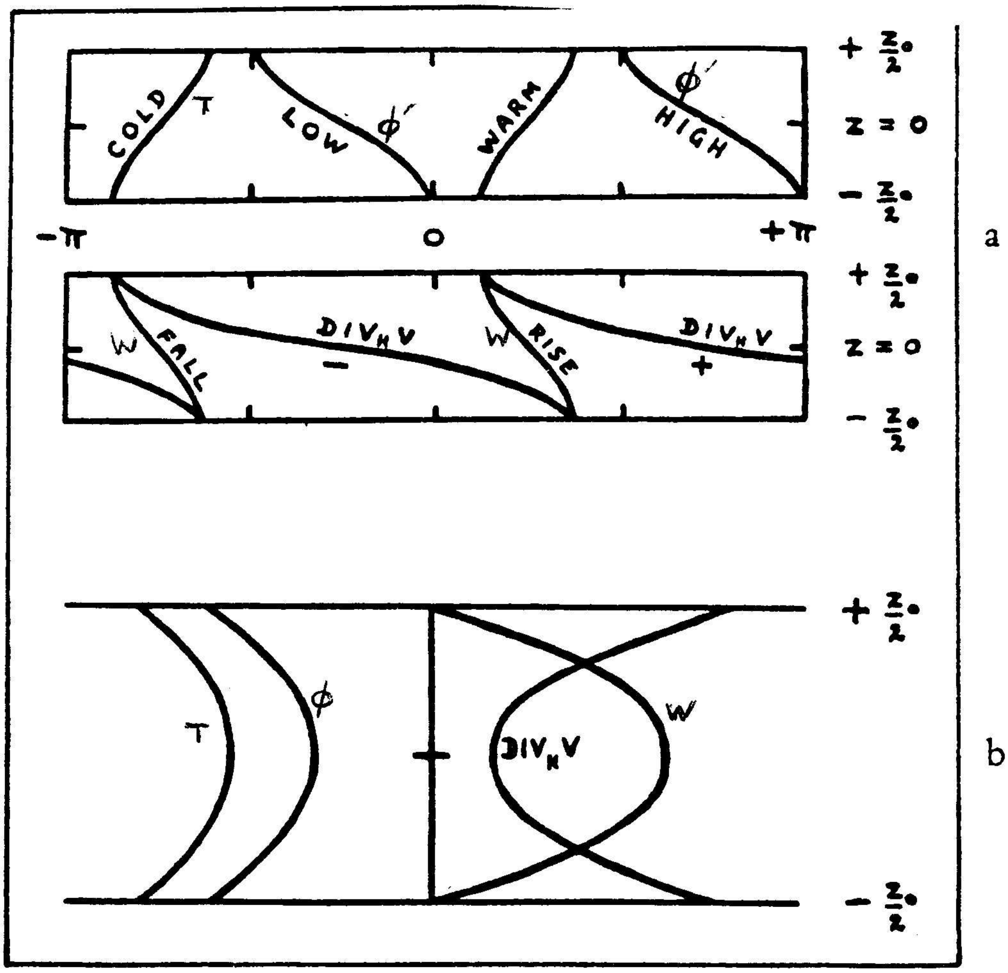
Putting in representative values of the Brunt-Vaisaila frequency, the Coriolis parameter and the depth of the troposphere, gives a wavelength of about 3000km, corresponding to e-folding times of about 1.5 or 2 days.

In his paper Eady noted that this implies some limit to our ability to predict the future state of the atmosphere. He reasoned that we only know the initial state of the atmosphere to a certain accuracy in the streamfunction $\delta\psi$, say and that the error will grow to be $\delta\psi \times \exp|\sigma_{\max}|t$, where $|\sigma_{\max}|$ is the maximum growth rate. This will eventually become unacceptably large.

Structure of waves

Solving for A and B in Eq 3for the growing waves and then computing the phases and amplitude and phases of the perturbations gives distributions as in the figures below. The phase stays constant expect for the fact that the whole structure moves to the east at the speed of the basic flow in the middle of the channel. In addition the relative amplitudes of the perturbations stay the same, but their absolute values increase exponentially in time as we have seen.

Some features to note in the phases is that the lines on which φ' reaches a maximum (or for that matter a minimum) slope to the west with increasing height.



From the definition of the streamfunction we see that to the east of the minimum in φ' as far as the next maximum, there is a northward component of velocity, while to the west of the minimum in φ' as far as the next maximum, there is a southward component of velocity. (These velocity components have not been indicated on the figure.) The fact that the phase of φ' slopes to the west with height implies – via the thermal wind relationship – that the air to the west of the minimum in φ' is cold, while that to the east is warm. This means that warm air is being advected northward by the perturbation and cold air southward. Near the boundaries, this horizontal advection is the only process which can produce the local changes in (potential) temperature. In the middle of the channel vertical velocities can affect it. We see from the phase of w that the warm air is rising and the cold air sinking. Consistently with that the magnitude of the temperature perturbation is smaller in the centre of the channel than at the boundaries.

From the continuity equation we must have horizontal convergence below middle level rise, and horizontal divergence above the rise. Thus the phases of $div_p\mathbf{v}$ have to be as shown.

If we travel with the velocity of the basic flow at the centre of the channel, the phases stay constant, and the basic flow relative to the structure is from west to east near the trop boundary and from east to west near the bottom boundary. Now maximum relative vorticity is found at minimum streamfunction (since $\zeta_{rel} = \nabla^2\psi' = -(\lambda^2 + \mu^2)\psi'$ for these waves). At the bottom air reaches the vorticity maximum from the east (seen in this moving frame), so its vorticity is increasing until it reaches the maximum. Consistently with this there is horizontal convergence in that air. As the air flows westwards out of the maximum vorticity it must be subjected to horizontal divergence to reduce the vorticity in time for the vorticity minimum. We see that this is indeed the case. At the top the distributions of vorticity change and horizontal convergence and divergence are also seen to be consistent, taking into account that at those heights the air is flowing through the system f rom west to east.

Slant convection

A brief section to be added here to complete these notes